M3031806F1PV1

M303/H

Module Examination 2018

Further Pure Mathematics

Friday 8 June 2018 **2.30 pm – 5.30 pm**

Time allowed: 3 hours

The exam is in two parts. Attempt *all* questions in Part 1 (each question is worth 11 marks) and *two* out of the three questions in Part 2 (each question is worth 17 marks). If you submit attempts to all three questions in Part 2, your best two scores will count towards your result.

It should take about two hours to do the Part 1 questions and an hour to do two Part 2 questions.

Start each question on a new page.

When sketching sets, take care to indicate which points are included: use dashed lines to indicate boundaries that are not included and solid lines to indicate boundaries that are. Use small open circles $\circ$ to indicate any 'corner' points on a boundary that are not included and small closed circles $\bullet$ to indicate any 'corner' points that are included.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Put all your used answer books together with your signed desk record on top. Fasten them in the top left corner with the round paper fastener. Attach this question paper to the back of the answer books with the flat paper clip.

The use of calculators is *not* permitted in this examination.

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PART 1

Attempt all questions. Each question is worth 11 marks. All of the Part 1 questions count towards your result.

Question 1

(a) (i) Use the Euclidean algorithm to find $\text{hcf}(147, 336)$. [3]

(ii) Hence find integers s and t such that

$$147s + 336t = \text{hcf}(147, 336).$$ [3]

(b) Prove by induction that

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for all positive integers n. [5]

Question 2

(a) Let $G = \mathbb{Z}_3 \times \mathbb{Z}_4$.

(i) Write down the order of G. [1]

(ii) Write down the order of $(0, 3)$. [1]

(iii) Write down an element of order 6. [1]

(iv) Is G cyclic? Justify your answer. [2]

(b) Consider the following two groups, G and H, both of which have order 1200.

$$G = \mathbb{Z}_{24} \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$$
$$H = \mathbb{Z}_{12} \times \mathbb{Z}_{20} \times \mathbb{Z}_5$$

(i) Write each of G and H as a direct product of cyclic groups of prime power order (and hence deduce that G and H are not isomorphic groups). [2]

(ii) Is either of G or H isomorphic to $\mathbb{Z}_{1200}$? Briefly justify your answer. [2]

(iii) Is either of G or H isomorphic to D_{600}? Briefly justify your answer. [2]

Question 3

(a) Given $60 = 2^2 \times 3 \times 5$, list all the divisors of 60.

Hence, or otherwise, find

(i) $\tau(60)$,

(ii) $\sigma(60)$,

(iii) $\phi(60)$. [3]

(b) Calculate the following Legendre symbols (note that 179 and 223 are prime)

(i) $(2/223)$,

(ii) $(179/223)$. [4]

(c) With the help of the Rational Root Test, prove that

$$f(x) = x^3 - x^2 + 3x - 2$$

is irreducible over $\mathbb{Q}$. [4]

Question 4

Let

$$A = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 2\},\ B = \{(a, b) \in \mathbb{R}^2 : |b| < |a|\}$$

and

$$C = A \cap B.$$

(a) Sketch on separate diagrams or describe in words each of A, B and C taking care to indicate which points lie in the sets. [3]

(b) Determine whether $(1, 0)$ is an interior point of C for the Euclidean metric $d^{(2)}$. [2]

(c) Write down in set notation the closure, interior and boundary of C for the Euclidean metric $d^{(2)}$ on the plane. [4]

(d) Show that the closure of C for the Euclidean metric on the plane is compact. [2]

Question 5

Let $\alpha = \sqrt{6} - \sqrt{10}$; you may assume that
$[\mathbb{Q}(\sqrt{6}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{10}) : \mathbb{Q}] = 2$, $\sqrt{6} \notin \mathbb{Q}(\sqrt{10})$ and $\sqrt{10} \notin \mathbb{Q}(\sqrt{6})$.

(a) Express α^3 in the form $a\sqrt{6} + b\sqrt{10}$ for suitable $a, b \in \mathbb{Q}$. [2]

(b) With the help of (a) or otherwise, show that $\sqrt{6}, \sqrt{10} \in \mathbb{Q}(\alpha)$. [3]

(c) Hence show that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{6}, \sqrt{10})$. [1]

(d) Determine the degree $[\mathbb{Q}(\alpha) : \mathbb{Q}]$. [2]

(e) By considering α^2 find the minimal polynomial for α over $\mathbb{Q}$. [3]

Question 6

Define a metric, d, on $X = \mathbb{R}^2 - \{(a, b) : a^2 + b^2 = 2\}$ by

$$d(\mathbf{x}, \mathbf{y}) = \max\{2|x_1 - y_1|, |x_2 - y_2|\}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ are points in $\mathbb{R}^2$.
(You do not have to show that d is a metric.)

(a) Write down $d\big((0, 1), (1, 0)\big)$. [*1*]

(b) Write down a d-disconnection of X. [*2*]

(c) Show that the sequence $((\frac{1}{n}, \frac{\sin n}{n}))_{n=1}^{\infty}$ is a d-convergent sequence in X. [*3*]

(d) By considering the sequence $\big((\sqrt{2} + \frac{1}{n}, 0)\big)_{n=1}^{\infty}$, show that X is not d-complete. [*4*]

(e) Deduce that X is not d-sequentially compact. [*1*]

PART 2

Attempt two questions. Each question is worth 17 marks. (If you submit attempts to all three questions, your best two scores will count towards your result.)

Question 7

Let G be a group of order $539 = 11 \times 7^2$.

(a) (i) Write down the order of the Sylow 7-subgroup. [*1*]

(ii) Show that G has a unique Sylow 11-subgroup and a unique Sylow 7-subgroup. [*3*]

(iii) Deduce that G is abelian. [*5*]

(b) The dicyclic group of order 40 is given by the following presentation:

$$\mathrm{Dic}_{10} = \langle a, b : a^4 = e,\ a^2 = b^{10},\ aba^3b = e \rangle .$$

Express $abab^3$ in the form $a^i b^j$, for $i = 0, 1, \ldots, 3$ and $j = 0, 1, \ldots, 9$. [*3*]

(c) Let G be a finite abelian group and let m be a natural number that divides $|G|$. Let p be a prime that divides m.

(i) Show that G has a normal subgroup of order p. [*2*]

(ii) By using the Second Principle of Mathematical Induction on the order of $|G|$, the Correspondence Theorem and part (i) (or otherwise), prove that G has a subgroup of order m. [*3*]

Question 8

Let $N(a+bi) = a^2 + b^2$ be the usual Euclidean norm on the ring of Gaussian integers $\mathbb{Z}[i] = \{a+bi : a, b \in \mathbb{Z}\}$. (You may assume that this norm is multiplicative.)

(a) Show that the only elements $r \in \mathbb{Z}[i]$ for which $N(r) = 5$ are

$$1 \pm 2i,\ -1 \pm 2i,\ 2 \pm i \text{ and } -2 \pm i.$$

Explain why these elements can not all be associates of each other. [4]

(b) Let r be a common factor of $s, t \in \mathbb{Z}[i]$. Show that $N(r)$ is a common factor of $N(s)$ and $N(t)$ in $\mathbb{Z}$. [2]

(c) Hence show that $1 + 2i$ is an hcf of $3 + i$ and 5 in $\mathbb{Z}[i]$. [3]

Let $I = \langle 5, 3+i \rangle$ be the ideal in $\mathbb{Z}[i]$ generated by the elements 5 and $3 + i$. Since $\mathbb{Z}[i]$ is a principal ideal domain, the ideal I is principal.

(d) Show that $\{1+2i, -1-2i, -2+i, 2-i\}$ is the set of all elements $d \in \mathbb{Z}[i]$ such that $I = \langle d \rangle$. [3]

(e) With the help of the previous parts or otherwise, show that I is a maximal ideal in $\mathbb{Z}[i]$. [2]

(f) By part (d), I is a principal ideal in $\mathbb{Z}[i]$ generated by $2 - i$, that is, $I = \langle 2 - i \rangle$. Using the results of the previous parts or otherwise, show that $I = \{\alpha + \beta i : \alpha, \beta \in \mathbb{Z},\ \alpha + 2\beta \equiv 0 \pmod 5\}$. [3]

Question 9

(a) Define a distance function $d\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ as follows:

$$d(a,b) = \begin{cases} 0, & \text{if } a - b = 0, \\ 2, & \text{if } a - b \text{ is even and non-zero}, \\ 3, & \text{if } a - b \text{ is odd}. \end{cases}$$

(i) Write down $d(1, 5)$. [1]

(ii) Prove that d is a metric for $\mathbb{N}$ by showing that

i. d satisfies (M1). [2]

ii. d satisfies (M2). [3]

iii. d satisfies (M3). [5]

(b) Let
$A = \{f \in C[0,1] : |f(x) - f(y)| \leq \sqrt{|x-y|} \text{ for each } x, y \in [0,1]\}$
and $B = \{f \in C[0,1] : |f(x)| \leq 1 \text{ for } 0 \leq x \leq 1\}$.

(i) Show that A is $d_{\max}$-closed. [3]

(ii) Deduce that $A \cap B$ is $d_{\max}$-closed. [1]

(iii) Hence or otherwise show that $A \cap B$ is $d_{\max}$-sequentially compact. (You may assume that A is an equicontinuous set.) [2]

[END OF QUESTION PAPER]